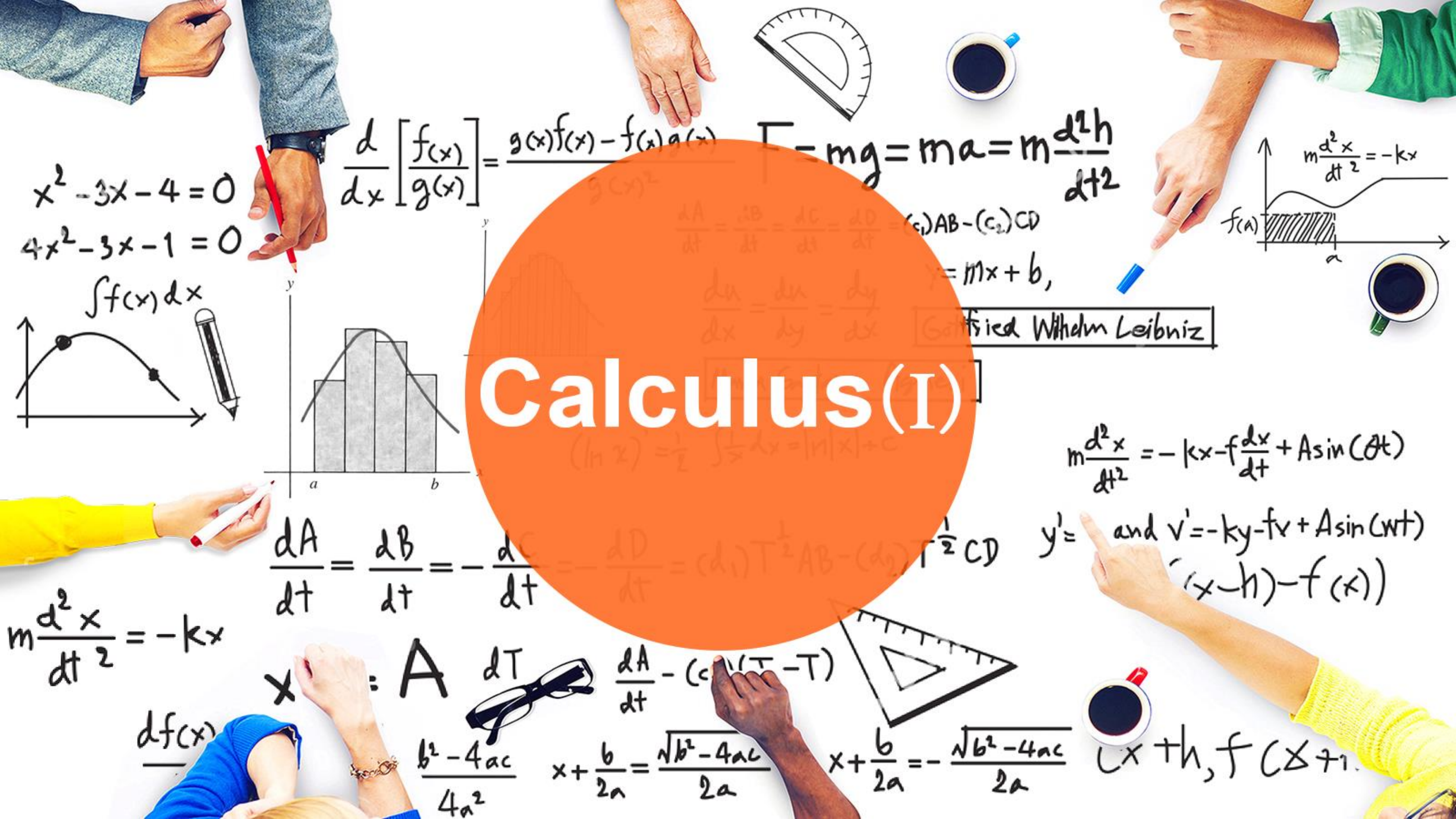
Calculus(I)



Integration of Rational Functions Using Partial Fractions

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How to integrate the function?

$$\int \frac{x^3 + x + 1}{x^2 + 1} dx$$?

$$\int \frac{x + 3}{x^2 - 5x + 6} dx$$?

$$\int \frac{1}{(1 + 2x)(1 + x^2)} dx$$?

Integration of Rational Functions

The division of two rational function

$$\frac{P(x)}{Q(x)} = \frac{a_0 x^{n} + a_1 x^{n-1} + \cdots + a_n}{b_0 x^{m} + b_1 x^{m-1} + \cdots + b_m}$$

m , n are all non-negative integers

there is no common factor between molecules and the denominator.

$(1)\, n < m,$ Real fraction

$(2)\, n \geq m,$ Improper fraction

Integration of Rational Functions

$$\frac{P(x)}{Q(x)} = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$$

Improper fraction

Radical function $\xlongequal[\text{division}]{}$ { **polynomials** / **real fraction** }

The sum of a number of the parts

Integration of Rational Functions

$$Q(x) = b_0(x-a)^{\lambda}\cdots(x^2+px+q)^{\mu}\cdots, (p^2-4q<0)$$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)^{\lambda}} + \frac{A_2}{(x-a)^{\lambda-1}} + \cdots + \frac{A_{\lambda}}{(x-a)^{1}} + \cdots$$

# of coefficients to be determined: λ

$$+\frac{M_1x+N_1}{(x^2+px+q)^{\mu}} + \frac{M_2x+N_2}{(x^2+px+q)^{\mu-1}} + \cdots + \frac{M_{\mu}x+N_{\mu}}{x^2+px+q} + \cdots$$

# of coefficients to be determined: 2μ

Example 1

Find $\int \frac{x^3+x+1}{x^2+1}dx$ 。 。

Improper fraction

$$\frac{x^3+x+1}{x^2+1} = x + \frac{1}{x^2+1}$$

$$\int \frac{x^3+x+1}{x^2+1}dx = \int x dx + \int \frac{dx}{x^2+1} = \frac{x^2}{2} + \arctan x + C$$

Tips: If the integrand function is improper fraction,

divide it into: a polynomial + a real fraction.

Example 2

Find $\int \frac{x+3}{x^2-5x+6}dx$

$$\frac{x+3}{x^2-5x+6}=\frac{x+3}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$$

factoring:

$$x+3=A(x-3)+B(x-2)$$

$$x+3=(A+B)x-(3A+2B)$$

$$\Rightarrow\begin{cases}A+B=1,\\ -(3A+2B)=3,\end{cases}\Rightarrow\begin{cases}A=-5\\ B=6\end{cases}$$

Comparing coefficient

$$\therefore \frac{x+3}{x^2-5x+6}=\frac{-5}{x-2}+\frac{6}{x-3}$$

Example 2

So, we have $\int \frac{x+3}{x^2-5x+6}dx$

$$= \int \left[\frac{-5}{x-2} + \frac{6}{x-3} \right] dx$$

$$= -5\int \frac{1}{x-2}dx + 6\int \frac{1}{x-3}dx$$

$$= -5\ln|x-2| + 6\ln|x-3| + C$$

Example 3

Find $\int \frac{1}{(1+2x)(1+x^2)}dx$

$$\frac{1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$$

quadratic formula factor

$$1 = A(1+x^2) + (Bx+C)(1+2x)$$

$$1 = (A+2B)x^2 + (B+2C)x + C + A$$

Comparing coefficient

$$\begin{cases} A+2B=0, \\ B+2C=0, \\ A+C=1, \end{cases} \Rightarrow A=\frac{4}{5}, B=-\frac{2}{5}, C=\frac{1}{5}$$

Example 3

So, we have $\therefore \dfrac{1}{(1+2x)(1+x^2)} = \dfrac{\frac{4}{5}}{1+2x} + \dfrac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}$

$$\int \frac{1}{(1+2x)(1+x^2)}dx = \int \frac{\frac{4}{5}}{1+2x}dx + \int \frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}dx$$

$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\int \frac{2x}{1+x^2}dx + \frac{1}{5}\int \frac{1}{1+x^2}dx$$

$$= \frac{2}{5}\ln|1+2x| - \frac{1}{5}\ln(1+x^2) + \frac{1}{5}\arctan x + C$$

Summary

Tip1: Improper fraction = **a polynomial + a real fraction**.

Tip2: $Q(x)=b_0(x-a)^\lambda\cdots(x^2+px+q)^\mu\cdots,\quad (p^2-4q<0)$

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(x-a)^\lambda}+\frac{A_2}{(x-a)^{\lambda-1}}+\cdots+\frac{A_\lambda}{(x-a)^1}+\cdots$$

# of coefficients to be determined: λ

$$+\frac{M_1x+N_1}{(x^2+px+q)^\mu}+\frac{M_2x+N_2}{(x^2+px+q)^{\mu-1}}+\cdots+\frac{M_\mu x+N_\mu}{x^2+px+q}+\cdots$$

# of coefficients to be determined: 2μ

Questions and Answers

$$\int \frac{5x+3}{x^3-2x^2-3x}dx$$

$$\text{Let } \frac{5x+3}{x(x-3)(x+1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

$$\Rightarrow A=-1, B=-\frac{1}{2}, C=\frac{3}{2},$$

$$=\int -\frac{1}{x}-\frac{1}{2(x+1)}+\frac{3}{2(x-3)}dx$$

$$=-\ln|x|-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-3|+C$$

Questions and Answers

$$\int \frac{6x^2-3x+1}{(4x+1)(x^2+1)}dx$$

Let $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)} = \dfrac{A}{4x+1} + \dfrac{Bx+C}{x^2+1}$ $\Rightarrow A=2, B=1, C=-1,$

$$I = \int \frac{2}{4x+1} + \frac{x-1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|4x+1| + \frac{1}{2}\int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|4x+1| + \frac{1}{2}\ln\left(x^2+1\right) - \arctan x + C$$

Questions and Answers

$$\int \frac{1}{x(x-1)^2}dx$$

Let $\dfrac{1}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-1}$

$\Rightarrow 1 = A(x-1)^2 + Bx + Cx(x-1)$

let $x = 0, \Rightarrow A = 1,$ let $x = 1, \Rightarrow B = 1$

let $x = 2, \Rightarrow C = -1$

$$\therefore \frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}$$

Questions and Answers

We have $\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}$ $\int \frac{1}{x(x-1)^2}dx$

So,

$$\int \frac{1}{x(x-1)^2}dx$$

$$= \int \left[\frac{1}{x} + \frac{1}{(x-1)^2} - \frac{1}{x-1}\right]dx$$

$$= \int \frac{1}{x}dx + \int \frac{1}{(x-1)^2}dx - \int \frac{1}{x-1}dx$$

$$= \ln|x| - \frac{1}{x-1} - \ln|x-1| + C$$

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